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Existence of solutions of vector variational inequalities and vector complementarity problems

Q. H. Ansari · A. P. Farajzadeh · S. Schaible

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Abstract In this paper, we consider vector variational inequality and vector F-complementarity problems in the setting of topological vector spaces. We extend the concept of upper sign continuity for vector-valued functions and provide some existence results for solutions of vector variational inequalities and vector F-complementarity problems. Moreover, the nonemptyness and compactness of solution sets of these problems are investigated under suitable assumptions. We use a version of Fan-KKM theorem and Dobrowolski's fixed point theorem to establish our results. The results of this paper generalize and improve several results recently appeared in the literature.

Keywords Vector variational inequalities · Vector F-complementarity problems · KKM mapping · C_x-pseudomonotonicity · C_x-upper sign continuity · Positively homogeneous mappings

1 Introduction

The theory of vector variational inequalities has been extensively studied in the last two decades because of its applications to vector optimization problems, vector complementarity

Q. H. Ansari
Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals,
P.O. Box 1169, Dhahran 31261, Saudi Arabia

Q. H. Ansari
Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
e-mail: qhansari@sancharnet.in

A. P. Farajzadeh (✉)
Department of Mathematics, Razi University, Kermanshah 67149, Iran
e-mail: ali-ff@sci.razi.ac.ir

S. Schaible
Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li 32023, Taiwan
e-mail: schaible2008@gmail.com

problems, game theory, economics, etc; See, for example, [6,7,11,12] and references therein. In the recent past, different kinds of monotonicities were introduced to study various kinds of (vector) variational inequalities and (vector) complementarity problems. For details, we refer to [2,5,8–10,14] and references therein. Chen [2] introduced the concept of semimonotonicity by combining the compactness and monotonicity, and studied the so-called semimonotone scalar variational inequality in the setting of Banach spaces. Recently, Fang and Huang [5,10] considered a more general vector variational inequality problem and extended the semimonotone scalar variational inequality to the vector case. They studied the existence of solutions of such problem with applications to vector complementarity problems.

Let X and Y be two topological vector spaces, K a nonempty convex subset of X, $C : K \to 2^Y$ a set-valued mapping with proper solid convex cone values, and $F : K \to Y$ a mapping, where 2^Y denotes the family of all subsets of Y. We denote by $L(X, Y)$ the space of all continuous linear operators from X into Y and by $\langle s, x\rangle$ the evaluation of $s \in L(X, Y)$ at $x \in X$. Let $T : K \to L(X, Y)$ be a given mapping. We consider the following Stampacchia-type vector variational inequality problem (in short, SVVIP): find $\bar{x} \in K$ such that

$$\langle T(\bar{x}), y - \bar{x}\rangle + F(y) - F(\bar{x}) \notin -\mathrm{int}C(\bar{x}), \quad \forall y \in K,$$

where $\mathrm{int}C(x)$ denotes the interior of $C(x)$. The set of solutions of SVVIP is denoted by S_S.

Another problem which is closely related to SVVIP is the following Minty-type vector variational inequality problem (in short, MVVIP): find $\bar{x} \in K$ such that

$$\langle T(y), y - \bar{x}\rangle + F(y) - F(\bar{x}) \notin -\mathrm{int}C(\bar{x}), \quad \forall y \in K.$$

We denote the set of solutions of MVVIP by S_M. These two problems have been considered and studied by Fang and Huang [5,10] for a fixed cone. They provided the existence of solutions of these problems under different kinds of pseudomonotonicities and hemicontinuity. They have also provided applications of these problems to vector F-complementarity problems.

We also considered the following more general Stampacchia-type vector variational inequality problem (in short, GSVVIP): find $\bar{x} \in K$ such that

$$\langle A(\bar{x}, \bar{x}), y - \bar{x}\rangle + F(y) - F(\bar{x}) \notin -\mathrm{int}C(\bar{x}), \quad \forall y \in K,$$

where $A : K \times K \to L(X, Y)$ is a mapping. Fang and Huang [5] considered this problem for a fixed cone and proved the existence of its solution under demipseudomonotonicity and hemicontinuity assumptions in the setting of reflexive Banach spaces. We note that the scalar version of the above mentioned problem was first considered and studied by Chen [2]. He established the existence of solutions of his problem under semi-monotonicity. As an application, Fang and Huang [5] proved the existence of solutions of the following vector F-complementarity problem: find $\bar{x} \in K$ such that

$$\langle A(\bar{x}, \bar{x}), \bar{x}\rangle + F(\bar{x}) \notin \mathrm{int}P \text{ and } \langle A(\bar{x}, \bar{x}), y\rangle + F(y) \notin -\mathrm{int}P, \quad \forall y \in K,$$

where $A : K \times K \to L(X, Y)$ and P is a proper solid convex cone in Y.

In this paper, we also consider the following more general vector F-complementarity problem (in short, GVCP): find $\bar{x} \in K$ such that

$$\langle A(\bar{x}, \bar{x}), \bar{x}\rangle + F(\bar{x}) \notin \mathrm{int}C(\bar{x}) \text{ and } \langle A(\bar{x}, \bar{x}), y\rangle + F(y) \notin -\mathrm{int}C(\bar{x}), \quad \forall y \in K.$$

In the next section, we introduce the concept of C_x-upper sign continuity which extend the previous concept of upper sign continuity introduced by Hadjisavvas [8]. We also recall some known definitions and results which will be used in the sequel. In Sect. 3, we establish the nonemptyness of the solution set of SVVIP under C_x-upper sign continuity with or without pseudomonotonicity assumptions. The last section deals with the existence of solutions of GSVVIP and GVCP under C_x-upper sign continuity in the setting of metrizable topological vector spaces but without demipseudomonotonicity assumption. We use the Dobrowolski's fixed point theorem and a version of the Fan-KKM theorem to extend and improve some results of Fang and Huang [5].

2 Preliminaries

Let X and Y be two topological vector spaces, K a nonempty convex subset of X and $C : K \to 2^Y$ a set-valued mapping such that for all $x \in K$, $C(x)$ is a proper closed convex cone with $\operatorname{int}C(x) \neq \emptyset$. Let $F : K \to Y$ be a mapping.

Recall, a mapping $T : K \to L(X, Y)$ is said to be *hemicontinuous* if, for any fixed $x, y \in K$, the mapping $t \mapsto \langle T(x + t(y - x)), y - x\rangle$ is continuous at 0^+.

Definition 1 Let $x \in K$ be any arbitrary element. The mapping $T : K \to L(X, Y)$ is said to be C_x-upper sign continuous with respect to F if, for all $y \in K$ and $t \in\ ]0, 1[$,

$$\begin{aligned}&\langle T(x + t(y - x)), y - x\rangle + F(y) - F(x) \notin -\operatorname{int}C(x), \quad \forall t \in\ ]0, 1[\\ &\quad \Rightarrow \langle T(x), y - x\rangle + F(y) - F(x) \notin -\operatorname{int}C(x).\end{aligned}$$

Remark 1 It is easy to see that the hemicontinuity of T implies C_x-upper sign continuity of T with respect to F. If $X = Y = \mathbb{R}$, $K = C(x) = [0, \infty)$ and $F \equiv 0$, then any positive mapping $T : K \to L(X, Y) \equiv \mathbb{R}$ is C_x-upper sign continuous while it is not hemicontinuous. In this case, the concept of C_x-upper sign continuity reduces to upper sign continuity introduced by Hadjisavvas [8].

Fang and Huang [5] defined the pseudomonotonicity of $T : K \to L(X, Y)$ with respect to F in the following manner: For any given $x, y \in K$,

$$\langle T(x), y - x\rangle + F(y) - F(x) \notin -\operatorname{int}P \ \Rightarrow\ \langle T(y), y - x\rangle + F(y) - F(x) \in P, \quad (2.1)$$

where P is a pointed solid closed convex cone in Y.

We point out that this definition of pseudomonotonicity is too strong. If $P = \mathbb{R}^n_+$, then condition (2.1) says that if one coordinate of $\langle T(x), y - x\rangle + F(y) - F(x)$ is nonnegative, then all coordinates of $\langle T(y), y - x\rangle + F(y) - F(x)$ are nonnegative. If we replace $\langle T(y), y - x\rangle + F(y) - F(x) \in P$ by $\langle T(y), y - x\rangle + F(y) - F(x) \notin -\operatorname{int}P$ in (2.1), then condition (2.1) would say that if one coordinate of $\langle T(y), y - x\rangle + F(y) - F(x)$ is nonnegative implies at least one coordinate of $\langle T(y), y - x\rangle + F(y) - F(x)$ is also nonnegative. Therefore, we adopt the following definition of pseudomonotonicity of T with respect to F.

Definition 2 Let $x \in K$ be any arbitrary element. A mapping $T : K \to L(X, Y)$ is said to be C_x-pseudomonotone with respect to F if, for all $y \in K$,

$$\langle T(x), y - x\rangle + F(y) - F(x) \notin -\operatorname{int}C(x) \Rightarrow \langle T(y), y - x\rangle + F(y) - F(x) \notin -\operatorname{int}C(x).$$

The following example shows that our definition of C_x-pseudomonotonicity w.r.t. F is more general than the one used by Fang and Huang [5,10].

Example 1 Let $X = K = \mathbb{R}$, $Y = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \quad y \geq 0\}$ be a fixed closed convex cone in Y. Let us define $T(x)(t) = \langle T(x), t\rangle = t(x, x^2)$ and $F(x) = 0$. Then, obviously, $T(x) \in L(X, Y)$.

If we take $y < x, x < 0$, then $\langle T(x), y-x\rangle = (y-x)(x, x^2) \notin -\mathrm{int}P$ since $(y-x)x > 0$ and $\langle T(y), y-x\rangle = (y-x)(y, y^2) \notin P$ because $(y-x)y^2 < 0$ and so T is not pseudo-monotone mapping in the sense of Huang and Fang [10]. While, if $\langle T(y), y-x\rangle \in -\mathrm{int}P$ then $(x-y)(y, y^2) \in \mathrm{int}P$. Thus $x-y > 0$ and $y > 0$ which imply that $\langle T(x), x-y\rangle = (x-y)(x, x^2) \in \mathrm{int}P$ and so $\langle T(x), y-x\rangle \in -\mathrm{int}P$. This shows that T is P-pseudomonotone with respect to F in our sense.

Rest of the paper, unless otherwise specified, $P = \bigcap_{x\in K} C(x)$ is a fixed proper solid convex cone in Y.

Definition 3 A mapping $F : K \to Y$ is said to be P-convex if,

$$F(x) + t(F(y) - F(x)) - F(x + t(y - x)) \in P, \quad \forall t \in [0, 1], \quad \forall x, y \in K.$$

It is easy to see that if F is P-convex then for all $x_i \in K$, $t_i \in]0, 1[$ for all $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n t_i = 1$, we have $\sum_{i=1}^n t_i F(x_i) - F\left(\sum_{i=1}^n t_i x_i\right) \in P$.

Definition 4 Let X and Y be two topological spaces. A set-valued mapping $T : X \to 2^Y$ is called:

(i) upper semi-continuous at $x \in X$ if, for each open set V containing $T(x)$, there is an open set U containing x such that for all $t \in U$, $T(t) \subset V$;
T is said to be upper semi-continuous on X if, it is upper semi-continuous at every point $x \in X$;
(ii) closed if, the graph $G_r(T) = \{(x, y) \in X \times Y : x \in X,\ y \in T(x)\}$ of T is a closed set;
(iii) compact if, the closure of range T, that is, $clT(X)$ is compact, where $T(X) = \bigcup_{x\in X} T(x)$.

Proposition 1 *[1] Let X and Y be two topological spaces. If $T : X \to 2^Y$ is closed and compact, then it is upper semi-continuous on X.*

Definition 5 [15] Let K be a nonempty subset of a topological space X. A set-valued mapping $\Gamma : K \to 2^K$ is said to be transfer closed-valued on K if, for all $x \in K$, $y \notin \Gamma(x)$ implies that there exists a point $x' \in K$ such that $y \notin cl_K\Gamma(x')$, where $cl_K\Gamma(x)$ denotes the closure of $\Gamma(x)$ in K.

It is well known that Γ is transfer closed-valued if and only if $\bigcap_{x\in K} cl_K\Gamma(x) = \bigcap_{x\in K} \Gamma(x)$.

Definition 6 Let K_0 be a nonempty subset of K. A set-valued mapping $\Gamma : K_0 \to 2^K$ is said to be a KKM map if, $coA \subseteq \bigcup_{x\in A} \Gamma(x)$ for very finite subset A of K_0, where co denotes the convex hull.

Lemma 1 *[4] Let K be a nonempty subset of a topological vector space X and $\Gamma : K \to 2^X$ be a KKM mapping with closed values. Assume that there exist a nonempty compact convex subset $D \subseteq K$ such that $B = \bigcap_{x\in D} \Gamma(x)$ is compact. Then $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.*

Theorem 1 *[3,13] Let K be a convex subset of a metrizable topological vector space X and $F : K \to 2^K$ be a compact upper semi-continuous set-valued mapping with nonempty closed convex values. Then F has a fixed point in K.*

3 Existence of solutions of SVVIP

Throughout this section, unless otherwise specified, X and Y are topological vector spaces, K is a nonempty convex subset of X and $C : K \to 2^Y$ is a set-valued mapping such that for all $x \in K$, $C(x)$ is a solid proper closed convex cone.

In order to present our existence results for a solution of SVVIP, we establish the following lemma.

Lemma 2 *Let $F : K \to Y$ be a P-convex mapping and $T : K \to L(X, Y)$ be C_x-upper sign continuous and C_x-pseudomonotone with respect to F. Then, the solution sets of MVVIP and SVVIP are equal.*

Proof By C_x-pseudomonotonicity of T with respect to F, every solution of SVVIP is a solution of MVVIP.

Conversely, let $\bar{x}$ be a solution of MVVIP. Then, for any given $y \in K$ and $t \in \,]0, 1[$ and by letting $y_t = \bar{x} + t(y - \bar{x})$, we have

$$t\langle T(y_t), y - \bar{x}\rangle + F(y_t) - F(\bar{x}) \notin -\mathrm{int}C(\bar{x}). \tag{1}$$

The P-convexity of F implies that

$$F(\bar{x}) + t(F(y) - F(\bar{x})) - F(y_t) \in P \subseteq C(\bar{x}), \quad \forall t \in [0, 1]. \tag{2}$$

By using (1) and (2), we get

$$t\,(\langle T(y_t), y - \bar{x}\rangle + F(y) - F(\bar{x})) \notin -\mathrm{int}C(\bar{x}), \quad \forall t \in \,]0, 1].$$

Since $Y \setminus (-\mathrm{int}C(\bar{x}))$ is a cone, we obtain

$$\langle T(y_t), y - \bar{x}\rangle + F(y) - F(\bar{x}) \notin -\mathrm{int}C(\bar{x}),$$

and thus the result follows from the C_x-upper sign continuity of T with respect to F. □

Remark 2 Lemma 2 can be viewed as a generalization of Minty lemma for vector variational inequalities but under C_x-upper sign continuity. In fact, Lemma 2 can be viewed as an improvement of Lemma 2.3 in [5] as we have assumed C_x-upper sign continuity instead of hemicontinuity and considered moving cone instead of a fixed cone.

We now establish an existence result for a solution of SVVIP under C_x-upper sign continuity.

Theorem 2 *Let $F : K \to Y$ be a P-convex mapping and for all $x \in K$, let $T : K \to L(X, Y)$ be C_x-pseudomonotone and C_x-upper sign continuous with respect to F. Assume that that the following conditions hold.*

(i) The set-valued mapping $y \mapsto \{x \in K : \langle T(y), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x)\}$ is transfer closed-valued on K.
(ii) There exist compact subset $B \subseteq K$ and compact convex subset $D \subseteq K$ such that $\forall x \in K \setminus B$, $\exists y \in D$ such that $\langle T(y), y - x\rangle + F(y) - F(x) \in -\mathrm{int}C(x)$.

Then the solution set S_S of SVVIP is nonempty and compact.

Proof For all $y \in K$, define a set-valued mapping $\Gamma : K \to 2^K$ as

$$\Gamma(y) = \{x \in K : \langle T(y), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x)\}.$$

We claim that Γ is a KKM map. Otherwise, there exist $y_1, \ldots, y_n \in K$ and $z \in \mathrm{co}(\{y_1, \ldots, y_n\})$ such that $z \notin \bigcup_{i=1}^n \Gamma(y_i)$. Then,

$$\langle T(y_i), y_i - z\rangle + F(y_i) - F(z) \in -\mathrm{int}C(z), \quad \text{for all } i = 1, 2, \ldots, n.$$

Since T is C_x-pseudomonotone with respect to F, we have

$$\langle T(z), y_i - z\rangle + F(y_i) - F(z) \in -\mathrm{int}C(z) \quad \text{for all } i = 1, 2, \ldots, n. \tag{3}$$

For each $i = 1, 2, \ldots, n$, let $t_i \in]0, 1[$ with $\sum_{i=1}^n t_i = 1$. Multiplying relation (3) by t_i and summing, we obtain

$$\sum_{i=1}^n t_i \langle T(z), y_i - z\rangle + \sum_{i=1}^n t_i F(y_i) - \sum_{i=1}^n t_i F(z) \in -\mathrm{int}C(z).$$

By P-convexity of F, we obtain

$$\left\langle T(z), \sum_{i=1}^n t_i y_i - z \right\rangle + F\left(\sum_{i=1}^n t_i y_i\right) - F(z) \in -\mathrm{int}C(z),$$

and thus $0 = \langle T(z), z - z\rangle + F(z) - F(z) \in -\mathrm{int}C(z)$ which contradicts to our assumption that $C(z) \neq Y$.

By condition (ii), $cl_K\left(\bigcap_{y\in D} \Gamma(y)\right) \subseteq B$. Consequently, set-valued mapping $cl\Gamma: K \to 2^K$ satisfies all the conditions of Lemma 1 and so $\bigcap_{x\in K} \Gamma(x)$ is nonempty. By condition (i), we get $S_S = \bigcap_{x\in K} cl\Gamma(x) = \bigcap_{x\in K} \Gamma(x)$ which implies that the solution set S_M of MVVIP is nonempty. Moreover, since T is C_x-upper sign continuous with respect to F and F is P-convex, by using Lemma 2, we get

$$S_S = \bigcap_{y\in K} \Gamma(y) = \bigcap_{y\in K} \{x \in K : \langle T(x), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x)\}.$$

This and conditions (i) and (ii) imply that the solution set of SVVIP is a nonempty and compact subset of B. □

Example 2 Let $X = \mathbb{R}$, $K = [0, 1]$, $Y = \mathbb{R}^2$ and $C(x) = P = \{(u, v) \in \mathbb{R}^2 : u \geq 0,\ v \geq 0\}$ for all $x \in K$, be a fixed closed convex cone in Y. Let us define $T(x)(t) = \langle T(x), t\rangle = t(x, x^2)$ and $F(x) = 0$ for all $x \in K$ and $t \in X$. Then, F is P-convex and T is C_x-pseudomonotone and C_x-upper sign continuous with respect to F and

$$\langle T(x), y - x\rangle + F(y) - F(x) = (y - x)(x, x^2) = ((y - x)x, (y - x)x^2).$$

It is easy to see that the set $\{x \in K : \langle T(y), y - x\rangle \notin -\mathrm{int}C(x)\} = [0, y]$ is closed and so the mapping $y \mapsto \{x \in K : \langle T(y), y - x\rangle \notin -\mathrm{int}C(x)\}$ is transfer closed valued on K. Since K is compact, condition (ii) of Theorem 2 trivially holds. Therefore, T satisfies all the assumptions of Theorem 2 and so the solution set of SVVIP is nonempty and compact. It is clear that only $x = 0$ satisfies the following relation

$$\langle T(x), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x), \quad \forall y \in K.$$

Similarly, only $x = 0$ satisfies the following relation

$$\langle T(y), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x), \quad \forall y \in K.$$

Hence the solution sets of SVVIP and MVVIP are equal to the singleton set $\{0\}$.

Remark 3

(a) If X is a real reflexive Banach space and K is a nonempty bounded closed convex subset of X, then K is weak* compact. In this case, condition (ii) of Theorem 2 can be removed.
(b) It is obvious that if F is continuous and the set-valued map $W(x) = Y\backslash(-\mathrm{int}C(x))$ for all $x \in K$, is closed, then condition (i) of Theorem 2 trivially holds.
(c) Theorem 2 can be seen as an improvement of Theorem 2.1 in [5] as we have assumed C_x-upper sign continuity instead of hemicontinuity and we also used coercivity condition (ii) instead of boundedness of K.

Now we prove the existence of a solution of SVVIP without any kind of pseudomonotonicity assumption.

Theorem 3 *Let K, X, Y and C be the same as in Theorem 2 and $F : K \to Y$ be a map. Assume that the set-valued mapping $T : K \to 2^K$ satisfies the following conditions.*

(i) For all $y \in K$, the set $\{x \in K : \langle T(x), y - x\rangle + F(y) - F(x) \notin -intC(x)\}$ is convex.
(ii) The set-valued mapping $y \mapsto \{x \in K : \langle T(x), y - x\rangle + F(y) - F(x) \notin -intC(x)\}$ is transfer closed-valued on K.
(iii) There exist compact subset $B \subseteq K$ and compact convex subset $D \subseteq K$ such that $\forall x \in K\backslash B$, $\exists y \in D$ such that $\langle T(x), y - x\rangle + F(y) - F(x) \in -intC(x)$.

Then the solution set S_S of SVVIP is nonempty and compact.

Proof For all $y \in K$, define $\Gamma : K \to 2^K$ as

$$\Gamma(y) = \{x \in K : \langle T(x), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x)\}.$$

By the same argument as in the proof of Theorem 2, it is easy to see that $cl_K\Gamma$ satisfies all the conditions of Lemma 1, hence $\bigcap_{x\in K} cl_K\Gamma(x) \neq \emptyset$. Since $S_S = \bigcap_{x\in K} \Gamma(x)$, condition (ii) implies that S_S is nonempty and again by conditions (ii) and (iii), S_S is compact. □

Remark 4 Condition (ii) of Theorem 3 holds when F is continuous and the mapping $W(x) = Y\backslash(-\mathrm{int}C(x))$ is closed.

Example 3 Let $X = Y = \mathbb{R}$, $K = [0, 1]$, $F(x) = x$, $C(x) = [0, \infty)$ for all $x \in K$. We define $T : K \to L(X, Y) = \mathbb{R}$ by

$$T(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

It is easy to see that T is C_x-upper sign continuity with respect to F (note that T is a non-negative mapping and F is continuous) while T is not upper semicontinuous (if x is an irrational number and $\{x_n\}$ is a sequence of rational numbers in $[0, 1]$, then the relation $\limsup T(x_n) \leq T(x)$ does not hold). For all $y \in K$, we have

$$\{x \in [0, 1] : \langle T(x), y - x\rangle + F(y) - F(x) \notin -\mathrm{int}C(x)\} = [0, y]$$

which is closed and convex. Then T satisfies all the conditions of Theorem 3 and so the solution set of SVVIP is nonempty and compact.

We claim that the solution set of SVVIP is the singleton set $\{0\}$. If x is a rational number belongs to [0,1] and a solution, then the following relation does not hold

$$\langle Tx, y - x\rangle + F(y) - F(x) = F(y) - F(x) = y - x \notin -\mathrm{int}C(x), \ \forall y \in K = [0, 1].$$

Similarly, if $x \in (0, 1]$ is a rational number then the previous relation also does not hold.

Finally, if $x = 0$ then $\langle Tx, y - x\rangle + F(y) - F(x) = F(y) - F(x) = 2y \notin -\mathrm{int}C(x)$ for all $y \in K = [0, 1]$ holds. Similarly, we can easily see that the solution set of MVVIP is the singleton set $\{0\}$.

4 Existence of solutions of GSVVI

Now we establish the following existence result for a solution of GSVVI under C_x-pseudomonotonicity and C_x-upper sign continuity but without demipseudomonotonicity assumption. This theorem generalizes and improves Theorem 3.1 in [5].

Theorem 4 *Let K be a nonempty closed convex subset of a metrizable topological vector space X and $C : K \to 2^Y$ be a set-valued mapping such that for all $x \in K$, $C(x)$ is a proper solid convex cone. Let $F : K \to Y$ be a P-convex and continuous mapping and $W : K \to 2^Y$ be a closed set-valued mapping defined as $W(x) = Y\setminus(-\mathrm{int}C(x))$ for all $x \in K$ such that $tW(x) + (1-t)W(y) \subseteq W(tx + (1-t)y)$ for all $x, y \in K$ and $t \in [0, 1]$. Let $A : K \times K \to L(X, Y)$ be a mapping. Assume that the following conditions hold:*

(i) For all $z \in K$, the mapping $A(\cdot, z) : K \to L(X, Y)$ is finite-dimensional continuous, that is, for any finite dimensional subspace $M \subseteq X$, $A(\cdot, z) : K \cap M \to L(X, Y)$ is continuous;
(ii) A is C_x-pseudomonotone and C_x-upper sign continuous in the second argument;
(iii) For each finite dimensional subspace M of X with $K_M = K \cap M \neq \emptyset$, there exist compact subset $B_M \subseteq K_M$ and compact convex subset $D_M \subseteq K_M$ such that $\forall(x, z) \in K_M \times (K_M\setminus B_M)$, $\exists y \in D_M$ such that $\langle A(x, z), y - z\rangle + F(y) - F(z) \in -\mathrm{int}C(z)$.

Then GSVVIP has a solution.

Proof Let $M \subset X$ be a finite dimensional subspace with $K_M = K \cap M \neq \emptyset$. For each fixed $w \in K$, consider the problem of finding $\bar{u} \in K_M$ such that

$$\langle A(w, \bar{u}), v - \bar{u}\rangle + F(v) - F(\bar{u}) \notin -\mathrm{int}C(\bar{u}), \quad \forall v \in K_M. \tag{4}$$

By Theorem 2, the problem (4) has a nonempty compact solution set.

For all $w \in M$, define a set-valued mapping $T : K_M \to 2^{K_M}$ as

$$T(w) = \{u \in K_M : \langle A(w, u), v - u\rangle + F(v) - F(u) \notin -\mathrm{int}C(u), \quad \forall v \in K_M\}.$$

Then $T(w)$ is a nonempty closed subset of B_M, in fact, $T(w)$ is the solution set of (4) corresponding to w. By Lemma 2, we have

$$T(w) = \{u \in K_M : \langle A(w, v), v - u\rangle + F(v) - F(u) \notin -\mathrm{int}C(u), \quad \forall v \in K_M\}$$

which is a convex set.

Indeed, let $u_i \in T(w)$ for $i = 1, 2$, then for all $v \in K_M$

$$\langle A(w, v), v - u_i\rangle + F(v) - F(u_i) \notin -\mathrm{int}C(u_i), \quad \text{for } i = 1, 2,$$

that is,

$$\langle A(w, v), v - u_i\rangle + F(v) - F(u_i) \in W(u_i), \quad \text{for } i = 1, 2.$$

Multiplying this relation for $i = 1$ by t and for $i = 2$ by $(1 - t)$, where $t \in]0, 1[$ and then summing them, we obtain

$$\begin{aligned}&\langle A(w, v), v - (tu_1 + (1-t)u_2)\rangle + F(v) - (tF(u_1) + (1-t)F(u_2))\\&\quad \in tW(u_1) + (1-t)W(u_2) \subseteq W\left(tu_1 + (1-t)u_2\right),\end{aligned}$$

and therefore,

$$\langle A(w, v), v - (tu_1 + (1-t)u_2)\rangle + F(v) - (tF(u_1) + (1-t)F(u_2)) \\ \notin -\mathrm{int}C\,(tu_1 + (1-t)u_2)\,.$$

Since F is P-convex, we have

$$tF(u_1) + (1-t)F(u_2) - F(u_t) \in P \subseteq C(u_t), \quad \text{where } u_t = tu_1 + (1-t)u_2.$$

By combining last two relations, we get

$$\langle A(w, v), v - u_t\rangle + F(v) - F(u_t) \notin -\mathrm{int}C(u_t)$$

and thus $u_t \in T(w)$.

Since for each $v \in K_M$, the mapping $A(\cdot, v) : K_M \to L(X, Y)$ and F are continuous and W is closed, the graph of T

$$\mathrm{graph}(T) = \{(w, u) \in K_M \times K_M : \langle A(w, v), v - u\rangle + F(v) - F(u) \\ \notin -\mathrm{int}C(u), \quad \forall v \in K_M\}$$

is closed and therefore T is a closed map.

Since $T(K_M) = \bigcup_{w\in K_M} T(w) \subseteq B_M$, T is a compact map. Proposition 1 implies that T is upper semicontinuous. Theorem 1 entails that T has a fixed point $\bar{w} \in K_M$, that is,

$$\langle A(\bar{w}, \bar{w}), v - \bar{w}\rangle + F(v) - F(\bar{w}) \notin -\mathrm{int}C(\bar{w}), \quad \forall v \in K_M. \tag{5}$$

Set $\mathcal{M} = \{M \subset X : M \text{ is a finite dimensional subspace with } K_M \neq \emptyset\}$ and for $M \in \mathcal{M}$

$$W_M = \{u \in K : \langle A(u, v), v - u\rangle + F(v) - F(u) \notin -\mathrm{int}C(u), \ \forall v \in K_M\}.$$

Since $A(\cdot, w)$ is continuous on K_M, F is continuous on K and W is closed, we have W_M is closed. By (5), W_M is a nonempty subset of a compact set B_M. Therefore, W_M is nonempty and closed subset of a compact set B_M and hence it is nonempty and compact.

For each finite subset $\{M_i\}_{i=1}^n$ of $\mathcal{M}$, from the definition of W_M, we have $W_{\bigcup_i M_i} \subset \bigcap_{i=1}^n W_{M_i}$, so $\{W_M : M \in \mathcal{M}\}$ has the finite intersection property. Hence, there exists $u \in \bigcap_{M\in\mathcal{M}} W_M$.

We claim that

$$\langle A(u, u), v - u\rangle + F(v) - F(u) \notin -\mathrm{int}C(u), \quad \forall v \in K.$$

Indeed, for each $v \in K$, let $M \in \mathcal{M}$ be such that $v \in K_M$ and $u \in K_M$. Since W_M is closed and $u \in W_M$, there exists a net $\{u_\alpha\} \subset W_M$ such that u_α converges to u. By the definition of W_M, we have

$$\langle A(u_\alpha, v), v - u_\alpha\rangle + F(v) - F(u_\alpha) \notin -\mathrm{int}C(u_\alpha).$$

The continuity of $A(\cdot, w)$ and F and closedness of W imply that

$$\langle A(u, v), v - u\rangle + F(v) - F(u) \notin -\mathrm{int}C(u), \quad \forall v \in K.$$

Hence by Lemma 2, we have

$$\langle A(u, u), v - u\rangle + F(v) - F(u) \notin -\mathrm{int}C(u), \quad \forall v \in K.$$

As an application of Theorem 4, we derive the existence result for a solution of GVCP.

□

Theorem 5 *Let C, W, F and A be the same as in Theorem 4 and let K be a nonempty closed convex cone in a metrizable topological vector space. Assume that all the conditions of Theorem 4 hold such that $F\left(\frac{1}{2}x\right) = \frac{1}{2}F(x)$ for all $x \in K$. Then GVCP has a solution.*

Proof By Theorem 4, there exists $\bar{x} \in K$ such that

$$\langle A(\bar{x}, \bar{x}), y - \bar{x}\rangle + F(y) - F(\bar{x}) \notin -\text{int}C(\bar{x}), \quad \forall y \in K. \tag{6}$$

Since $F(0) = \frac{1}{2}F(0)$, we have $F(0) - \frac{1}{2}F(0) = 0$ and so $F(0) = 0$. Letting $y = 0$ in (6), we obtain

$$\langle A(\bar{x}, \bar{x}), \bar{x}\rangle + F(\bar{x}) \notin \text{int}C(\bar{x}). \tag{7}$$

Substituting $y = \bar{x} + z$ into (6) for all $z \in K$, we deduce that

$$F(\bar{x}) - \langle A(\bar{x}, \bar{x}), z\rangle - F(\bar{x} + z) \notin \text{int}C(\bar{x}). \tag{8}$$

Since F is P-convex mapping, by multiplying the relation

$$\frac{1}{2}\left(F(\bar{x}) + F(z)\right) - F\left(\frac{1}{2}(\bar{x} + z)\right) \in P \subseteq C(\bar{x})$$

by 2 and using $F(\frac{1}{2}\bar{x}) = \frac{1}{2}F(\bar{x})$ for all $x \in K$, we get

$$F(\bar{x}) - F(\bar{x} + z) + F(z) \in C(\bar{x}). \tag{9}$$

By (8) and (9), we get

$$\langle A(\bar{x}, \bar{x}), z\rangle + F(z) \notin -\text{int}C(\bar{x}).$$

Because z was arbitrary element of K, we get the conclusion. □

Remark 5 The condition $F\left(\frac{1}{2}x\right) = \frac{1}{2}F(x)$ for all $x \in K$ holds if F is positively homogeneous, that is, $F(tx) = tF(x)$ for all $t \geq 0$. Hence, Theorem 5 generalizes and improves Theorem 3.2 in [5].

Finally, we give an example of a function F which satisfies the condition $F\left(\frac{1}{2}x\right) = \frac{1}{2}F(x)$ for all $x \in K$ of Theorem 5 but not a positively homogeneous function and hence Theorem 3.2 in [5] can not be applied.

Example 4 Let $F : \mathbb{R} \to \mathbb{R}$ be defined as

$$F(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then F satisfies the condition $F\left(\frac{1}{2}x\right) = \frac{1}{2}F(x)$ for all $x \in \mathbb{R}$ but it is not positively homogeneous.

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